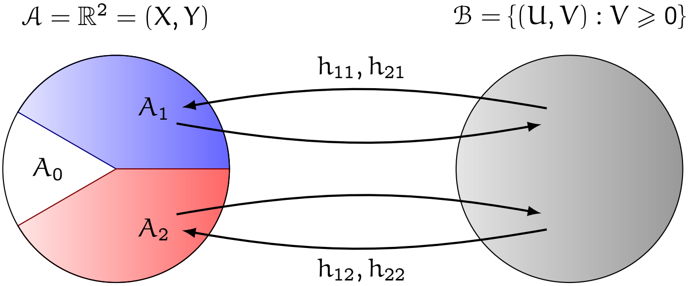
The ratio of two Gaussian random variables $X, Y \sim N(0,1)$ is a Cauchy distribution. Let $U = X/Y$ and $V = |Y|$. If $Y = 0$, U and V can be any value because $\Pr(Y=0) = 0$. If we restrict consideration to either positive or negative value of Y, then the transform from $\mathcal{A} = (X, Y)$ to $\mathcal{B} = (U, V)$ is one-to-one.

$$A_1 = \{(x,y) : y > 0\}, \qquad A_2 = \{(x,y) : y < 0\}, \qquad A_3 = \{(x,y) : y = 0\}$$

These three partition $\mathcal{A} = \mathbb{R}^2$ and $\Pr[(X,Y) \in A_0] = \Pr[Y = 0] = 0$.



$$\begin{aligned} \mathcal{B} \to \mathcal{A}_1 &: x = h_{11}(u,v) = uv, \quad & y = h_{21}(u,v) = v \\ \mathcal{B} \to \mathcal{A}_2 &: x = h_{12}(u,v) = -uv, \quad & y = h_{22}(u,v) = -v \end{aligned}$$

$$J_1 = \begin{vmatrix} \frac{\partial uv}{\partial u} & \frac{\partial uv}{\partial v} \\ \frac{\partial v}{\partial u} & \frac{\partial v}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v, \quad J_2 = \begin{vmatrix} \frac{\partial(-uv)}{\partial u} & \frac{\partial(-uv)}{\partial v} \\ \frac{\partial(-v)}{\partial u} & \frac{\partial(-v)}{\partial v} \end{vmatrix} = \begin{vmatrix} -v & -u \\ 0 & -1 \end{vmatrix} = v$$

$$f_{XY}(x,y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) = \frac{1}{2\pi} \exp\left(-\frac{x^2+y^2}{2}\right)$$

$$\begin{aligned} f_{UV}(u,v) &= \sum_{i=0}^{2} f_{XY}(h_{1i}(u,v), h_{2i}(u,v))|J_i| \\ &= \frac{1}{2\pi} \exp\left(-\frac{(uv)^2 + v^2}{2}\right)|v| + \frac{1}{2\pi} \exp\left(-\frac{(-uv)^2 + (-v)^2}{2}\right)|v| \\ &= \frac{v}{\pi} \exp\left(-\frac{v^2(u^2+1)}{2}\right) \end{aligned}$$

$$\begin{aligned} f_U(u) &= \int_0^\infty \frac{v}{\pi} \exp\left(-\frac{v^2(u^2+1)}{2}\right) dv \qquad & \text{(change of variable: } z = v^2\text{)} \\ &= \int_0^\infty \frac{1}{2\pi} \exp\left(-\frac{u^2+1}{2} z\right) dz \qquad & \left(\int_0^\infty \exp(-\alpha z) dz = 1/\alpha\right) \\ &= \frac{1}{\pi} \cdot \frac{1}{u^2+1} \end{aligned}$$