The Cauchy distribution is a symmetric distribution on $(-\infty, \infty)$ with pdf

$$f_X(x;\theta,\gamma) = \frac{1}{\pi} \cdot \frac{\gamma}{(x-\theta)^2 + \gamma^2}$$

In this paper, we only deal with the case $\theta = 0$.

Consider two independent Gaussian random variables $X, Y \sim N(0,1)$. We will prove that the ratio X/Y is a Cauchy distribution by (1) defining the transformation $U = X/Y$ and $V = |Y|$, (2) finding the joint pdf $F_{U,V}(u,v)$, and (3) integrating out V to obtain the marginal pdf of U.

Unfortunately, the mapping $U = X/Y$ and $V = |Y|$ is not one-to-one: the two points (x,y) and $(-x,-y)$ map to the same (u,v) We need to partition (X,Y) into A_0, A_1, A_2 such that the mapping from A_i to (U,V) is one-to-one.

1. $A_0 = \{(X,Y) : Y = 0\}$: This exceptional case does not happen because $\Pr[Y = 0] = 0$ when $Y \sim N(0,1)$.
2. $A_1 = \{(X,Y) : Y > 0\}$: The mapping $U = X/Y, V = |Y|$ is one-to-one, and the inverse mappings are $h_{11}(u,v) = uv$, $h_{21} = v$.
3. $A_2 = \{(X,Y) : Y < 0\}$: The mapping $U = X/Y, V = |Y|$ is one-to-one, and the inverse mappings are $h_{12}(u,v) = -uv$, $h_{22} = -v$.

$$\begin{aligned}
J_1 &= \begin{vmatrix} \frac{\partial h_{11}}{\partial u} & \frac{\partial h_{11}}{\partial v} \\ \frac{\partial h_{21}}{\partial u} & \frac{\partial h_{21}}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial uv}{\partial u} & \frac{\partial uv}{\partial v} \\ \frac{\partial v}{\partial u} & \frac{\partial v}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v \\
J_2 &= \begin{vmatrix} \frac{\partial h_{12}}{\partial u} & \frac{\partial h_{12}}{\partial v} \\ \frac{\partial h_{22}}{\partial u} & \frac{\partial h_{22}}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial(-uv)}{\partial u} & \frac{\partial(-uv)}{\partial v} \\ \frac{\partial(-v)}{\partial u} & \frac{\partial(-v)}{\partial v} \end{vmatrix} = \begin{vmatrix} -v & -u \\ 0 & -1 \end{vmatrix} = v
\end{aligned}$$

$$f_{X,Y}(x,y) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) \frac{1}{\sqrt{2\pi}} \exp(-y^2/2) = \frac{1}{2\pi} \exp\left(-\frac{x^2+y^2}{2}\right)$$

$$\begin{aligned}
f_{UV}(u,v) &= f_{XY}(h_{11}(u,v), h_{21}(u,v))|J_1| + f_{XY}(h_{12}(u,v), h_{22}(u,v))|J_2| \\
&= \frac{1}{2\pi} \exp\left(-\frac{(uv)^2+v^2}{2}\right)|v| + \frac{1}{2\pi} \exp\left(-\frac{(-uv)^2+(-v)^2}{2}\right)|v| \\
&= \frac{v}{\pi} \exp\left(-\frac{v^2(u^2+1)}{2}\right), \qquad -\infty < u < \infty, \quad 0 < v < \infty
\end{aligned}$$

$$\begin{aligned}
f_U(u) &= \int_0^\infty \frac{v}{\pi} \exp\left(-\frac{v^2(u^2+1)}{2}\right) dv && \text{integrating out } V \\
&= \int_0^\infty \frac{1}{2\pi} \exp\left(-\frac{u^2+1}{2} z\right) dz && \text{Let } z = v^2 \text{ and } dz = 2v dv \\
&= \frac{1}{2\pi} \cdot \frac{2}{u^2+1} && \int_0^\infty \exp(-\alpha z) dz = \frac{1}{\alpha} \\
&= \frac{1}{\pi} \cdot \frac{1}{u^2+1}, \qquad -\infty < u < \infty
\end{aligned}$$